Exam

26/01/2023, 8:30 am - 10:30 am

Instructions:

- Prepare your solutions in an **ordered, clear and clean way**. Avoid delivering solutions with scratches.
- Write your name and student number in **all** pages of your solutions.
- Clearly indicate each exercise and the corresponding answer. Provide your solutions with as much detail as possible.
- Use different pieces of paper for solutions of different exercises.
- Read first the whole exam, and make a strategy for which exercises you attempt first. Start with those you feel comfortable with!

Exercise 1: (1 point) Prove that $f(x,y) = (e^x + e^y, e^x + e^{-y})$ is locally invertible at every point $(x,y) \in \mathbb{R}^2$. Moreover, if $f(a) = b$, what is the derivative of f^{-1} at b?

Solution: The Jacobian of f is

$$J(x,y) = \begin{bmatrix} e^x & e^y \\ e^x & -e^{-y} \end{bmatrix}. \tag{1}$$

It follows that $\det J = -(e^{x-y} + e^{x+y})$, which is nonzero for all $(x,y) \in \mathbb{R}^2$. Thus, f is continuously differentiable, and its derivative is invertible for all $(x,y) \in \mathbb{R}^2$. It follows from the inverse function theorem (slide 5, lecture 3) that f is locally invertible at every point $(x,y) \in \mathbb{R}^2$.

Let $a = (a_1, a_2)$ so that $f(a_1, a_2) = (e^{a_1} + e^{a_2}, e^{a_1} + e^{-a_2}) = (b_1, b_2) = b$. From our previous argument we know that there is a function f^{-1} such that $f^{-1}(b_1, b_2) = (a_1, a_2)$. From slide 8, lecture 3, it follows that

$$\mathrm{D}f^{-1}(b) = \begin{bmatrix} e^{a_1} & e^{a_2} \\ e^{a_1} & -e^{-a_2} \end{bmatrix}^{-1}. \tag{2}$$

Exercise 2: (1.5 points) Consider the ODE $x'' + 3x' + 2x = \dfrac{1}{1+e^t}$.

a) Find the general solution of the given ODE.

b) Make a sketch of the vector field corresponding to the homogeneous equation $x'' + 3x' + 2x = 0$.

c) Is there a relationship between the vector field of part b) and the general solution of part a)?

Solution:

a) We start by rewriting the given ODE as:

$$X' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{1+e^t} \end{bmatrix}, \tag{3}$$

where $(X_1, X_2) = (x, x')$. The homogeneous equation has solution $X_h(t) = c_1 e^{-2t} u + c_2 e^{-t} v$, where u, v denote the corresponding eigenvectors:

- for u: $Au = -2u \to \begin{bmatrix} u_2 \\ -2u_1 - 3u_2 \end{bmatrix} = \begin{bmatrix} -2u_1 \\ -2u_2 \end{bmatrix} \to u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
- for v: $Av = -v \to \begin{bmatrix} v_2 \\ -2v_1 - 3v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix} \to v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

In other words, the solution to the homogeneous problem is given by

$$X_h(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{-2t} & e^{-t} \\ -e^{-2t} & -e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{4}$$

Hence, according to slide 9 of lecture 7, the fundamental matrix is $M(t) = \begin{bmatrix} 2e^{-2t} & e^{-t} \\ -e^{-2t} & -e^{-t} \end{bmatrix}$. It follows that $M(t)^{-1} = \begin{bmatrix} e^{2t} & e^{2t} \\ -e^{t} & -2e^{t} \end{bmatrix}$. Next we follow slide 9 again to compute:

$$\int_0^t M(s)^{-1} b(s) \mathrm{d}s = \int_0^t \begin{bmatrix} \dfrac{e^{2s}}{1+e^s} \\ \dfrac{-2e^s}{1+e^s} \end{bmatrix} \mathrm{d}s = \int_0^t \begin{bmatrix} \dfrac{u-1}{u} \mathrm{d}u \\ \dfrac{-2e^s}{1+e^s} \mathrm{d}s \end{bmatrix} = \begin{bmatrix} 1 + e^t - \ln(1+e^t) \\ -2\ln(1+e^t) \end{bmatrix} \tag{5}$$

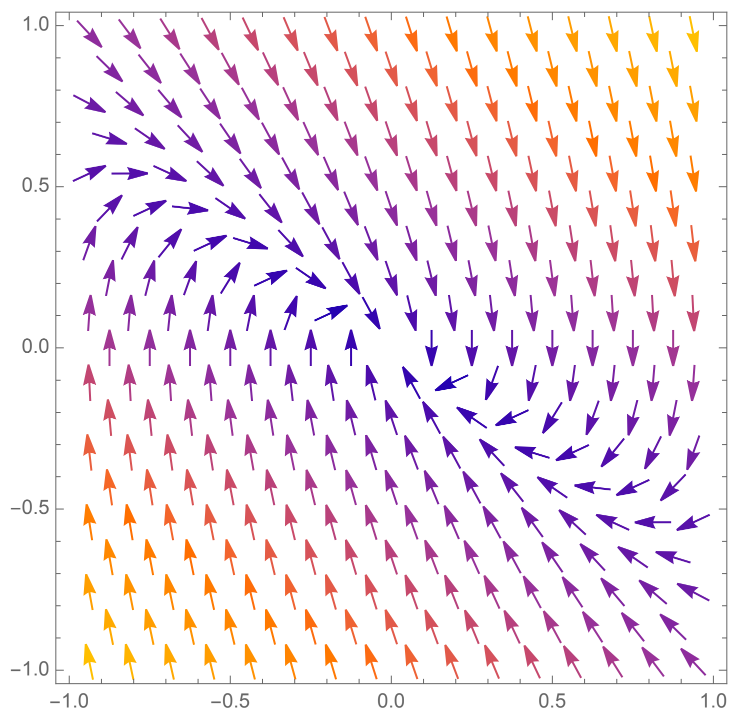
where we have made the change of variable $u = 1 + e^s$. Notice that we have also disregarded the constants because we only look for a particular solution, and such constants can otherwise be absorbed by the c_1 and c_2 constants of the homogeneous solution. So now we can write the full solution as

$$X(t) = \begin{bmatrix} 2e^{-2t} & e^{-t} \\ -e^{-2t} & -e^{-t} \end{bmatrix} \left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 1 + e^t - \ln(1+e^t) \\ -2\ln(1+e^t) \end{bmatrix} \right). \tag{6}$$

Finally, the solution $x(t)$ corresponds to the first component of $X(t)$. After relabeling the constants we thus obtain:

$$x(t) = c_1 e^{-2t} + c_2 e^{-t} + (e^{-2t} + e^{-t}) \ln(1+e^t). \tag{7}$$

b) The vector field looks like:



c) In principle no. However, one can expect that there is not much difference between the solutions of the homogeneous problem and the full problem since the particular solution vanishes exponentially fast.

Exercise 3: (1 point) Consider $f(t) = \int_t^{t^2} \frac{\mathrm{d}s}{s + \sin s}$ for $t > 1$. Compute the derivative of f.

Hint: it may be useful to write f as the composition of two functions, one of which is $(x, y) \mapsto \int_x^y \frac{\mathrm{d}s}{s + \sin s}$.

Solution: Using the hint, let $g(x, y) = \int_x^y \frac{\mathrm{d}s}{s + \sin s}$ and $h(t) = (t, t^2)$. Therefore $f(t) = (g \circ h)(t)$. We can now use the chain rule to compute the derivative. Notice that:

$$\begin{aligned} \mathrm{D}_1 g(x, y) &= -\frac{1}{x + \sin x} \\ \mathrm{D}_2 g(x, y) &= \frac{1}{y + \sin y} \\ \mathrm{D}h(t) &= \begin{bmatrix} 1 \\ 2t \end{bmatrix} \end{aligned} \tag{8}$$

where the first two equalities follow from the fundamental theorem of calculus in one variable. So, following slide 12 of lecture 1 we have:

$$f'(t) = \begin{bmatrix} -\frac{1}{t + \sin t} & \frac{1}{t^2 + \sin t^2} \end{bmatrix} \begin{bmatrix} 1 \\ 2t \end{bmatrix} = -\frac{1}{t + \sin t} + \frac{2t}{t^2 + \sin t^2}. \tag{9}$$

Exercise 4: (1 point) What is the n-dimensional volume of the region

$$\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i = 1, \dots, n \text{ and } x_1 + 2x_2 + \cdots + nx_n \leq n\}?$$

Solution: Dealing with the coefficients is a bit annoying, so let $y_i := \frac{n}{i} x_i$. Then the given region is alternatively given by

$$R_y = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n \,|\, y_i \geq 0 \text{ for all } i = 1, \dots, n \text{ and } y_1 + y_2 + \cdots + y_n \leq 1\}. \tag{10}$$

Since what we are doing is applying a linear transformation, it follows from slide 9 of lecture 9 that the required volume will be given by $\frac{n}{1} \cdot \frac{n}{2} \cdots \frac{n}{n} \operatorname{vol}_n R_y = \frac{n^n}{n!} \operatorname{vol}_n R_y$. So now we just compute the n-dimensional volume of R_y.

- Let $n = 1$. Then $\operatorname{vol}_1 R_y = 1$.
- Let $n = 2$. Then $\operatorname{vol}_2 R_y = \int_0^1 \int_0^{1-x_1} \mathrm{d}x_2 \mathrm{d}x_1 = \int_0^1 (1 - x_1) \mathrm{d}x_1 = \left(x_1 - \frac{x_1^2}{2} \right) \Big|_0^1 = \frac{1}{2}$.
- Let $n = 3$. Then

$$\begin{aligned} \operatorname{vol}_3 R_y &= \int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = \int_0^1 \int_0^{1-x_3} (1 - x_2 - x_3) \mathrm{d}x_2 \mathrm{d}x_3 \\ &= \int_0^1 \left(1 - x_3 - \frac{(1 - x_3)^2}{2} - x_3(1 - x_3) \right) \mathrm{d}x_3 \\ &= \int_0^1 \left(\frac{1}{2} - x_3 + \frac{x_3^2}{2} \right) \mathrm{d}x_3 = \left(\frac{1}{2} x_3 - \frac{x_3^2}{2} + \frac{x_3^3}{6} \right) \Big|_0^1 = \frac{1}{6}. \end{aligned} \tag{11}$$

It is thus safe to assume that $\operatorname{vol}_n R_y = \frac{1}{n!}$. Therefore, the volume of the given region is $\frac{n^n}{(n!)^2}$.

The proof by induction of the above argument is not necessary to get full points, we provide it here for completeness:

For $n \geq 1$, let A_n denote the n-volume of R_y. From the formulas above we see that we can write:

$$A_n = \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} \cdots \int_0^{1-x_1-\cdots-x_{n-1}} \mathrm{d}x_n \cdots \mathrm{d}x_2 \mathrm{d}x_1.$$

Let us define

$$B_n(t) = \int_0^t \int_0^{t-x_1} \int_0^{t-x_1-x_2} \cdots \int_0^{t-x_1-\cdots-x_{n-1}} \mathrm{d}x_n \cdots \mathrm{d}x_2 \mathrm{d}x_1.$$

Then

$$A_n = B_n(1).$$

From above it follows that $B_1(t) = t$, $B_2(t) = \frac{t^2}{2}$, $B_3(t) = \frac{t^3}{6}$. A natural guess is that $B_n(t) = \frac{t^n}{n!}$. One can show this by induction: assume that $B_n(t) = \frac{t^n}{n!}$, and notice that

$$B_{n+1}(t) = \int_0^t B_n\left(t - x_1\right) \mathrm{d}x_1.$$

It follows that:

$$B_{n+1}(t) = \int_0^t B_n\left(t - x_1\right) dx_1 = -\left[\frac{(t-x_1)^{n+1}}{(n+1)!}\right]_0^t = \frac{t^{n+1}}{(n+1)!}.$$

Thus, indeed $\boxed{A_n = B_n(1) = \frac{1}{n!}}$.

Exercise 5: (1 point) Let $\mathcal{S}$ be a closed curve in $\mathbb{R}^2$ and $\mathcal{C}$ the unit circle in $\mathbb{R}^2$. Suppose that $\mathcal{S}$ and $\mathcal{C}$ are diffeomorphic. What is the 2-dimensional volume of the curve $\mathcal{S}$ ($\mathrm{vol}_2\, \mathcal{S}$)? Justify your answer in full detail.

Hints and remarks: we are asking for the 2-volume, and not the 1-volume, of the 1-dimensional curve $\mathcal{S}$, and not of the region enclosed by it; for this exercise you may assume that "$\mathcal{S}$ and $\mathcal{C}$ are diffeomorphic" means that there is a C^r-function f, $r \geq 1$, with C^r inverse, such that $f : \mathcal{S} \to \mathcal{C}$ and $f^{-1} : \mathcal{C} \to \mathcal{S}$.

Solution: There are several ways to obtain the answer. The simplest is to recall that you already know that the 2-dimensional volume of the unit circle in $\mathbb{R}^2$ is zero. Since $\mathcal{S}$ is diffeomorphic to $\mathcal{C}$, then $\mathcal{S}$ is a 1-dimensional manifold in $\mathbb{R}^2$, and then use slide 6 of lecture 10 to conclude that $\mathrm{vol}_2\, \mathcal{S} = 0$. Another is to invoke the change of variables formula.

Exercise 6: (1.5 points) Let ω be the n-form in $\mathbb{R}^n$ defined by $\omega(e_1, \ldots, e_n) = 1$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^n$ given by $v_i = \sum_{1 \leq j \leq n} a_{ij} e_j$, where the a_{ij}'s are real scalars. Prove that

a) $\omega(v_1, \ldots, v_n) = \det A$, where $A = [a_{ij}]_{i,j=1,\ldots,n}$ is the $n \times n$ matrix with elements a_{ij}.

b) $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.

Solution:

a) It suffices to notice that $\omega(v_1, \ldots, v_n) = \omega(Ae_1, \ldots, Ae_n) = A^* \omega(e_1, \ldots, e_n)$, and to recall slide 13 of lecture 14 to prove the claim.

b) Notice that there is only one elementary n-form in $\mathbb{R}^n$. We then have that $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n(v_1, \ldots, v_n) = \det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \det A$, and hence equal to ω.

Exercise 7: (1 point) Let $z_1 = x_1 + \imath y_1$, $z_2 = x_2 + \imath y_2$ be coordinates in $\mathbb{C}^2$. Compute the integral of $\mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}y_1 \wedge \mathrm{d}x_2$ over the part of the locus of the equation $z_2 = z_1^k$ where $|z_1| < 1$, oriented by $\Omega = \operatorname{sgn} \mathrm{d}x_1 \wedge \mathrm{d}y_1$.

Solution: Let $r \in [0, 1)$ and $\theta \in [0, 2\pi]$. Then we define the parametrization

$$\gamma : (r, \theta) \mapsto (r\cos\theta, r\sin\theta, r^k\cos(k\theta), r^k\sin(k\theta)), \tag{12}$$

which indeed corresponds to the locus of $z_2 = z_1^k$. We have

$$\mathrm{D}_1\gamma = \begin{bmatrix} \cos\theta \\ \sin\theta \\ kr^{k-1}\cos(k\theta) \\ kr^{k-1}\sin(k\theta) \end{bmatrix}, \qquad \mathrm{D}_2\gamma = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ -kr^k\sin(k\theta) \\ kr^k\cos(k\theta) \end{bmatrix} \tag{13}$$

Notice that the parametrization is orientation preserving since $\operatorname{sgn} \mathrm{d}x_1 \wedge \mathrm{d}y_1(\mathrm{D}_1\gamma, \mathrm{D}_2\gamma) = r \geq 0$ and is only zero at the "bad point in the boundary" $r = 0$.

Next we have that

$$\begin{aligned}(\mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}y_1 \wedge \mathrm{d}x_2)(\mathrm{D}_1\gamma, \mathrm{D}_2\gamma) &= \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} + \det\begin{bmatrix} \sin\theta & r\cos\theta \\ kr^{k-1}\cos(k\theta) & -kr^k\sin(k\theta) \end{bmatrix} \\ &= r - kr^k(\sin(k\theta)\sin\theta + \cos(k\theta)\cos\theta).\end{aligned} \tag{14}$$

Therefore, the integral we are looking for is

$$\begin{aligned} I &= \int_0^1 \int_0^{2\pi} (r - kr^k(\sin(k\theta)\sin\theta + \cos(k\theta)\cos\theta))\mathrm{d}\theta\mathrm{d}r \\ &= \int_0^1 \int_0^{2\pi} r\mathrm{d}\theta\mathrm{d}r - \int_0^{2\pi} \int_0^1 kr^k(\sin(k\theta)\sin\theta + \cos(k\theta)\cos\theta)\mathrm{d}r\mathrm{d}\theta \\ &= \pi - \frac{kr^{k+1}}{k+1} \int_0^{2\pi} (\sin(k\theta)\sin\theta + \cos(k\theta)\cos\theta)\mathrm{d}\theta. \end{aligned} \tag{15}$$

Arriving to the previous expression is enough to get full points.

Solving the remaining integral gives +0.5 points, here we leave the solution:

- If $k = 0$, then $I = \pi$.
- If $k = 1$, then $I = \pi\left(1 - \frac{r^2}{2}\right)$
- If $k > 1$, then, again, $I = \pi$. The result follows immediately from the fact that, in this case, $\sin(k\theta)\sin\theta$ and $\cos(k\theta)\cos\theta$ are 2π-periodic with zero average.

Exercise 8: (1 point) Find the flux of the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r^a \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where a is a number and $r = \sqrt{x^2 + y^2 + z^2}$, through the surface S, where S is the sphere of radius R oriented by the outward-pointing normal.

Hint: the result is a function of a and R.

Solution: It is most convenient to parametrize the sphere of radius R using spherical coordinates, that is by $\gamma : (\theta, \phi) \mapsto (R\cos\theta\cos\phi, R\sin\theta\cos\phi, R\sin\phi)$ with $\theta \in [0, 2\pi]$ and $\phi \in [-\pi/2, \pi/2]$. To check if this is an orientation preserving parametrization we notice that

$$\mathrm{D}_1\gamma = \begin{bmatrix} -R\sin\theta\cos\phi \\ R\cos\theta\cos\phi \\ 0 \end{bmatrix}, \qquad \mathrm{D}_2\gamma = \begin{bmatrix} -R\cos\theta\sin\phi \\ -R\sin\theta\sin\phi \\ R\cos\phi \end{bmatrix}, \tag{16}$$

and therefore we have

$$\begin{aligned}
&\det\begin{bmatrix} R\cos\theta\cos\phi & -R\sin\theta\cos\phi & -R\cos\theta\sin\phi \\ R\sin\theta\cos\phi & R\cos\theta\cos\phi & -R\sin\theta\sin\phi \\ R\sin\phi & 0 & R\cos\phi \end{bmatrix} \\
&= R\sin\phi(R^2\sin^2\theta\sin\phi\cos\phi + R^2\cos^2\theta\cos\phi\sin\phi) + R\cos\phi(R^2\cos^2\theta\cos^2\phi + R^2\sin^2\theta\cos^2\phi) \\
&= R^3\sin\phi(\sin\phi\cos\phi) + R^3\cos\phi(\cos^2\phi) = R^3\cos\phi \geq 0.
\end{aligned} \tag{17}$$

Hence, indeed, the proposed parametrization is orientation preserving. Next, following slide 10 of lecture 13 we have:

$$\begin{aligned}
\int_S \Phi_{\vec{F}} &= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \det\begin{bmatrix} R^{a+1}\cos\theta\cos\phi & -R\sin\theta\cos\phi & -R\cos\theta\sin\phi \\ R^{a+1}\sin\theta\cos\phi & R\cos\theta\cos\phi & -R\sin\theta\sin\phi \\ R^{a+1}\sin\phi & 0 & R\cos\theta \end{bmatrix} \mathrm{d}\phi\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} R^{a+3}\cos\phi\mathrm{d}\phi\mathrm{d}\theta = 4\pi R^{a+3}.
\end{aligned} \tag{18}$$